

Twisted Fock Representations of Noncommutative Kähler Manifolds

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MSC 2010: 53D55 , 81R60

Abstract

We introduce twisted Fock representations of noncommutative Kähler manifolds and give their explicit expressions. The twisted Fock representation is a representation of the Heisenberg like algebra whose states are constructed by acting creation operators on a vacuum state. “Twisted” means that creation operators are not Hermitian conjugate of annihilation operators in this representation. In deformation quantization of Kähler manifolds with separation of variables formulated by Karabegov, local complex coordinates and partial derivatives of the Kähler potential with respect to coordinates satisfy the commutation relations between the creation and annihilation operators. Based on these relations, we construct the twisted Fock representation of noncommutative Kähler manifolds and give a dictionary to translate between the twisted Fock representations and functions on noncommutative Kähler manifolds concretely.

1 Introduction

Deformation quantization is a way to construct noncommutative geometry, which is first introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [3]. Several ways of deformation quantization were established by [9, 19, 10, 15]. In particular, deformation quantizations of Kähler manifolds were provided in [17, 18, 6, 7]. In this article, the deformation quantization with separation of variables is used to construct noncommutative Kähler manifolds that is introduced by Karabegov [11, 12, 14]. (For a recent review, see [25].) The deformation quantization is an

associative algebra on a set of formal power series of C^∞ functions with a star product between formal power series. One of the advantages of deformation quantization is that usual analytical techniques are available on noncommutative manifolds constructed in this way. On the other hand, when we consider field theories on noncommutative manifolds given by deformation quantization, physical quantities are given as formal power series, and there are difficulties to understand them from a viewpoint of physics. A typical way to solve the difficulties is to make a representation of the noncommutative algebra.

The purpose of this article is to construct the Fock representation of noncommutative Kähler manifolds. The algebras on noncommutative Kähler manifolds which are constructed by deformation quantization with separation of variables contain the Heisenberg like algebras. Local complex coordinates and partial derivatives of a Kähler potential satisfy the commutation relations between creation and annihilation operators. A Fock space is spanned by a vacuum, which is annihilated by all annihilation operators, and states obtained by acting creation operators on this vacuum. The algebras on noncommutative Kähler manifolds are represented as those of linear operators acting on the Fock space. We call the representation of the algebra the Fock representation. In representations studied in this article, creation operators and annihilation operators are not Hermitian conjugate with each other, in general. Therefore, the bases of the Fock space are not the Hermitian conjugates of those of the dual vector space. In this case, we call the representation the twisted Fock representation. Historically, Berezin constructed a kind of the Fock representations of some noncommutative Kähler manifolds[4, 5], and since then there have been various works on this subject [21, 25, 26, 20]. In this article, we construct the twisted Fock representation for an arbitrary noncommutative Kähler manifold given by deformation quantization with separation of variables [11, 12, 14].

One of the main results in this article is summarized as the following dictionary, Table 1. In this dictionary, z^i, \bar{z}^i ($i = 1, \dots, N$) are local complex coordinates of some open subset of an N dimensional Kähler manifold. Φ is a Kähler potential and H is defined by $e^{\Phi/\hbar} = \sum H_{\vec{m}, \vec{n}} z^{\vec{m}} \bar{z}^{\vec{n}}$, where $z^{\vec{m}} = z_1^{m_1} z_2^{m_2} \dots z_N^{m_N}$ for $\vec{m} = (m_1, m_2, \dots, m_N)$, and $\bar{z}^{\vec{n}}$ is similarly defined. a_i^\dagger and \underline{a}_i are essentially a creation operator and an annihilation operator, respectively. \underline{a}_i and \underline{a}_i^\dagger are Hermitian conjugate with each other. Note that a_i^\dagger is not a Hermitian conjugate of \underline{a}_i , in general. More detailed definitions are given in Section 2 and 3.

The twisted Fock algebra is defined on a local coordinate chart. The star product with separation variables are glued between charts with nonempty intersections. Therefore, transition functions between the twisted Fock algebras on two charts having an overlapping region are also constructed. Trace operations for the Fock representations as integrations of concerned functions are discussed. We observe

Table 1: Functions - Fock operators Dictionary

Functions	Fock operators
$e^{-\Phi/\hbar}$	$ \vec{0}\rangle\langle\vec{0} $
z_i	a_i^\dagger
$\frac{1}{\hbar}\partial_i\Phi$	\underline{a}_i
\bar{z}^i	$a_i = \sum \sqrt{\frac{\vec{m}!}{\vec{n}!}} H_{\vec{m},\vec{k}} H_{\vec{k}+\vec{e}_i,\vec{n}}^{-1} \vec{m}\rangle\langle\vec{n} $
$\frac{1}{\hbar}\partial_i\Phi$	$\underline{a}_i^\dagger = \sum \sqrt{\frac{\vec{m}!}{\vec{n}!}} (k_i + 1) H_{\vec{m},\vec{k}+\vec{e}_i} H_{\vec{k},\vec{n}}^{-1} \vec{m}\rangle\langle\vec{n} $

several examples, \mathbb{C}^N , a cylinder, $\mathbb{C}P^N$ and $\mathbb{C}H^N$.

The organization of this article is as follows. In Section 2, we review several facts of deformation quantization with separation of variables which are used in this article. In Section 3, a twisted Fock representation is constructed on a chart of a general Kähler manifold. In Section 4, transition maps between the twisted Fock representations on two local coordinate charts are constructed. In Section 5, we discuss a trace operation for the twisted Fock representation. In Section 6, the Fock representations of \mathbb{C}^N , a cylinder, $\mathbb{C}P^N$ and $\mathbb{C}H^N$ are given as examples. We summarize our results in Section 7.

2 A review of the deformation quantization with separation of variables

We give a general definition of deformation quantization, before moving into the deformation quantization for Kähler manifolds.

Definition 1 (Deformation quantization (weak sense)). Let M be a Poisson manifold. \mathcal{F} is defined as a set of formal power series:

$$\mathcal{F} := \left\{ f \mid f = \sum_k f_k \hbar^k, f_k \in C^\infty(M) \right\}. \quad (2.1)$$

Deformation quantization is defined as a structure of associative algebra of \mathcal{F} whose product is defined by a star product. The star product is defined as

$$f * g = \sum_k C_k(f, g) \hbar^k \quad (2.2)$$

such that the product satisfies the following conditions.

1. $*$ is associative product.
2. C_k is a bidifferential operator.
3. C_0 and C_1 are defined as

$$C_0(f, g) = fg, \quad (2.3)$$

$$C_1(f, g) - C_1(g, f) = i\{f, g\}, \quad (2.4)$$

where $\{f, g\}$ is the Poisson bracket.

4. $f * 1 = 1 * f = f$.

Note that this definition of deformation quantization is weaker than the usual definition of deformation quantization. The difference between them is in (2.4). In the strong sense of deformation quantization the condition $C_1(f, g) = \frac{i}{2}\{f, g\}$ is required.

As a special case of deformation quantizations of Kähler manifold M , deformation quantization with separation of variables is introduced by Karabegov [11, 12, 14].

Definition 2 (A star product with separation of variables). $*$ is called a star product with separation of variables when

$$a * f = af \quad (2.5)$$

for a holomorphic function a and

$$f * b = fb \quad (2.6)$$

for an anti-holomorphic function b .

The deformation quantization defined by using such a star product is also denoted deformation quantization with separation of variables. In this article, we consider only this type of deformation quantization for Kähler manifolds.

Let M be an N -dimensional complex Kähler manifold, Φ be its Kähler potential and ω be its Kähler 2-form:

$$\begin{aligned} \omega &:= ig_{k\bar{l}} dz^k \wedge d\bar{z}^l, \\ g_{k\bar{l}} &:= \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}. \end{aligned} \quad (2.7)$$

Here g is the Kähler metric and z^i, \bar{z}^j ($i, j = 1, \dots, N$) are local coordinates on an open set $U \subset M$ which is diffeomorphic to a connected open subset of \mathbb{C}^N . In this

paper, we use the Einstein summation convention over repeated indices. The $g^{\bar{k}l}$ is the inverse of the metric $g_{k\bar{l}}$:

$$g^{\bar{k}l} g_{l\bar{m}} = \delta_{\bar{k}\bar{m}}. \quad (2.8)$$

In the following, we use the following abridged notations

$$\partial_k = \frac{\partial}{\partial z^k}, \quad \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z}^k}. \quad (2.9)$$

Karabegov constructed a star product with separation of variables for Kähler manifolds in terms of differential operators [11, 12], as briefly explained below. For the left star multiplication by $f \in \mathcal{F}$, there exists a differential operator L_f such that

$$L_f g = f * g. \quad (2.10)$$

L_f is given as a formal power series in \hbar

$$L_f = \sum_{n=0}^{\infty} \hbar^n A^{(n)}, \quad (2.11)$$

where $A^{(n)}$ is a differential operator which contains only partial derivatives by z^i ($i = 1, \dots, N$) and has the following form

$$A^{(n)} = \sum_{k \geq 0} a_{\bar{i}_1 \dots \bar{i}_k}^{(n;k)} D^{\bar{i}_1} \dots D^{\bar{i}_k}, \quad (2.12)$$

where

$$D^{\bar{i}} = g^{\bar{i}j} \partial_j, \quad (2.13)$$

and each $a_{\bar{i}_1 \dots \bar{i}_k}^{(n;k)}$ is a C^∞ function on M . In particular, $a^{(n;0)}$ acts as a multiplication operator. Note that the differential operators $D^{\bar{i}}$ satisfy the following relations,

$$[D^{\bar{i}}, D^{\bar{j}}] = 0, \quad (2.14)$$

$$[D^{\bar{i}}, \partial_{\bar{j}} \Phi] = \delta_{ij}. \quad (2.15)$$

Karabegov showed the following theorem.

Theorem 2.1 (Karabegov[11, 12]). *L_f is uniquely determined by requiring the following conditions,*

$$L_f 1 = f * 1 = f, \quad (2.16)$$

$$[L_f, \partial_i \Phi + \hbar \partial_i] = 0, \quad (2.17)$$

This star product $*$ satisfies the associative condition

$$h * (g * f) = (h * g) * f. \quad (2.18)$$

Here is a useful theorem given by Karabegov.

Theorem 2.2 (Karabegov[11, 12]). *The differential operator L_f for an arbitrary function f is obtained from the operator $L_{\bar{z}^i}$, which corresponds to the left $*$ multiplication of \bar{z}^i ,*

$$L_f = \sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial}{\partial \bar{z}} \right)^{\alpha} f(L_{\bar{z}} - \bar{z})^{\alpha}, \quad (2.19)$$

where α is a multi-index.

Similarly, the differential operator $R_f = \sum_{n=0}^{\infty} \hbar^n B^{(n)}$ corresponding to the right $*$ multiplication by a function f contains only partial derivatives by \bar{z}^i and is determined by the conditions

$$R_f 1 = 1 * f = f, \quad (2.20)$$

$$[R_f, \partial_i \Phi + \hbar \partial_i] = 0. \quad (2.21)$$

$B^{(n)}$ has the following form,

$$B^{(n)} = \sum_{k \geq 0} b_{i_1 \dots i_k}^{(n; k)} D^{i_1} \dots D^{i_k}, \quad (2.22)$$

where $D^i = g^{i\bar{j}} \partial_{\bar{j}}$ and $b_{i_1 \dots i_k}^{(n; k)} \in C^{\infty}(M)$. The differential operator R_f for an arbitrary function f is obtained from the operator R_{z^i} , which corresponds to the right $*$ multiplication by z^i ,

$$R_f = \sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial}{\partial z} \right)^{\alpha} f(R_z - z)^{\alpha}. \quad (2.23)$$

In particular, the left star product by $\partial_i \Phi$ and the right star product by $\partial_{\bar{i}} \Phi$ are respectively written as

$$L_{\partial_i \Phi} = \hbar \partial_i + \partial_i \Phi = \hbar e^{-\Phi/\hbar} \partial_i e^{\Phi/\hbar}, \quad (2.24)$$

$$R_{\partial_{\bar{i}} \Phi} = \hbar \partial_{\bar{i}} + \partial_{\bar{i}} \Phi = \hbar e^{-\Phi/\hbar} \partial_{\bar{i}} e^{\Phi/\hbar}. \quad (2.25)$$

From the definition of the star product, we easily find

$$\left[\frac{1}{\hbar} \partial_i \Phi, z^j \right]_* = \delta_{ij}, \quad [z^i, z^j]_* = 0, \quad [\partial_i \Phi, \partial_j \Phi]_* = 0, \quad (2.26)$$

$$[\bar{z}^i, \frac{1}{\hbar} \partial_{\bar{j}} \Phi]_* = \delta_{ij}, \quad [\bar{z}^i, \bar{z}^j]_* = 0, \quad [\partial_i \Phi, \partial_{\bar{j}} \Phi]_* = 0, \quad (2.27)$$

where $[A, B]_* = A*B - B*A$. Hence, $\{z^i, \partial_j \Phi \mid i, j = 1, 2, \dots, N\}$ and $\{\bar{z}^i, \partial_j \Phi \mid i, j = 1, 2, \dots, N\}$ constitute $2N$ sets of the creation and annihilation operators under the star product. But, it should be noted that operators in $\{z^i, \partial_j \Phi\}$ does not commute with ones in $\{\bar{z}^i, \partial_j \Phi\}$, e.g., $z^i * \bar{z}^j - \bar{z}^j * z^i \neq 0$.

3 The Fock representation of noncommutative Kähler manifolds

In this section we introduce the Fock space on an open set $U \subset M$ which is diffeomorphic to a connected open subset of \mathbb{C}^N and an algebra as a set of linear operators acting on the Fock space.

As mentioned in Section 2, from the (2.26) and (2.27) $\{z^i, \partial_j \Phi \mid i, j = 1, 2, \dots, N\}$ and $\{\bar{z}^i, \partial_j \Phi \mid i, j = 1, 2, \dots, N\}$ are candidates for the creation and annihilation operators under the star product $*$. We introduce $a_i^\dagger, a_i, \underline{a}_i^\dagger$ and \underline{a}_i ($i = 1, 2, \dots, N$) by

$$a_i^\dagger = z^i, \quad \underline{a}_i = \frac{1}{\hbar} \partial_i \Phi, \quad a_i = \bar{z}^i, \quad \underline{a}_i^\dagger = \frac{1}{\hbar} \partial_i \Phi. \quad (3.1)$$

Then they satisfy the following commutation relations which are similar to the usual commutation relations for the creation and annihilation operators but slightly different,

$$[\underline{a}_i, a_j^\dagger]_* = \delta_{ij}, \quad [a_i^\dagger, a_j^\dagger]_* = 0, \quad [\underline{a}_i, \underline{a}_j]_* = 0, \quad (3.2)$$

$$[a_i, \underline{a}_j^\dagger]_* = \delta_{ij}, \quad [\underline{a}_i^\dagger, \underline{a}_j^\dagger]_* = 0, \quad [a_i, a_j]_* = 0. \quad (3.3)$$

There are differences from ordinary creation and annihilation operators that these two sets of creation and annihilation operators are not given as direct sum, in other words,

$$[a_i, a_i^\dagger]_* \quad \text{and} \quad [\underline{a}_i, \underline{a}_i^\dagger]_* \quad (3.4)$$

do not vanish in general.

The star product with separation of variables has the following property under the complex conjugation.

Proposition 3.1.

$$\overline{f * g} = \overline{L_f g} = \bar{g} * \bar{f} \quad (3.5)$$

Proof. As described in the previous section, L_f and R_f are uniquely determined by the equations (2.16), (2.17), (2.20), and (2.21). From the complex conjugation of (2.16) and (2.17), we find

$$\overline{L_f}1 = \bar{f}, \quad [\overline{L_f}, \partial_i \Phi + \hbar \partial_i] = 0. \quad (3.6)$$

Because of the uniqueness of solution of (2.20) and (2.21), $\overline{L_f}$ is equal to $R_{\bar{f}}$.

$$\overline{f * g} = \overline{L_f g} = \overline{L_f} \bar{g} = R_{\bar{f}} \bar{g} = \bar{g} * \bar{f}. \quad (3.7)$$

□

The Fock space is defined by a vector space spanned by the bases which is generated by acting a_i^\dagger on $|\vec{0}\rangle$,

$$\begin{aligned} |\vec{n}\rangle &= |n_1, \dots, n_N\rangle \\ &= c_1(\vec{n})(a_1^\dagger)^{n_1} * \dots * (a_N^\dagger)^{n_N} * |\vec{0}\rangle, \end{aligned} \quad (3.8)$$

where $|\vec{0}\rangle = |0, \dots, 0\rangle$ satisfies

$$\underline{a}_i * |\vec{0}\rangle = 0 \quad (i = 1, \dots, N) \quad (3.9)$$

and $(A)_*^n$ stands for $\overbrace{A * \dots * A}^n$. $c_1(\vec{n})$ is a normalization coefficient which does not depend on z^i and \bar{z}^i . Here, we define the basis of a dual vector space by acting \underline{a}_i on $\langle \vec{0} |$,

$$\begin{aligned} \langle \vec{m} | &= \langle m_1, \dots, m_N | \\ &= \langle \vec{0} | * (\underline{a}_1)^{m_1} * \dots * (\underline{a}_N)^{m_N} c_2(\vec{m}), \end{aligned} \quad (3.10)$$

and

$$\langle \vec{0} | * a_i^\dagger = 0 \quad (i = 1, \dots, N), \quad (3.11)$$

where $c_2(\vec{m})$ is also a normalization constant. The underlines are attached to the bra vectors in order to emphasize that $\langle \vec{m} |$ is not Hermitian conjugate to $|\vec{m}\rangle$. In this article, we set the normalization constants as

$$c_1(\vec{n}) = \frac{1}{\sqrt{\vec{n}!}}, \quad c_2(\vec{n}) = \frac{1}{\sqrt{\vec{n}!}}, \quad (3.12)$$

where $\vec{n}! = n_1! n_2! \dots n_N!$.

Definition 3. The local twisted Fock algebra (representation) F_U is defined as a algebra given by a set of linear operators acting on the Fock space defined on U :

$$F_U := \left\{ \sum_{\vec{n}, \vec{m}} A_{\vec{n}\vec{m}} |\vec{n}\rangle \langle \vec{m}| \mid A_{\vec{n}\vec{m}} \in \mathbb{C} \right\}. \quad (3.13)$$

and products between its elements are given by the star product $*$.

In the remaining part of this section, we construct concrete expressions of functions which are elements of this local twisted Fock algebra.

Lemma 3.2 (Berezin). *For arbitrary Kähler manifolds (M, ω) , there exists a Kähler potential $\Phi(z^1, \dots, z^N, \bar{z}^1, \dots, \bar{z}^N)$ such that*

$$\Phi(0, \dots, 0, \bar{z}^1, \dots, \bar{z}^N) = 0, \quad \Phi(z^1, \dots, z^N, 0, \dots, 0) = 0. \quad (3.14)$$

This is easily shown as follow. If a Kähler potential Φ satisfying $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi$ does not satisfy (3.14), then we redefine a new Kähler potential Φ' as

$$\begin{aligned} & \Phi'(z^1, \dots, z^N, \bar{z}^1, \dots, \bar{z}^N) \\ & := \Phi(z^1, \dots, z^N, \bar{z}^1, \dots, \bar{z}^N) - \Phi(0, \dots, 0, \bar{z}^1, \dots, \bar{z}^N) - \Phi(z^1, \dots, z^N, 0, \dots, 0). \end{aligned} \quad (3.15)$$

$\Phi(z^1, \dots, z^N, 0, \dots, 0)$ is a holomorphic function and $\Phi(0, \dots, 0, \bar{z}^1, \dots, \bar{z}^N)$ is an anti-holomorphic function. Kähler potentials have ambiguities of adding holomorphic and anti-holomorphic functions. This Φ' satisfies the condition (3.14). In the following, we abbreviate $\Phi(z^1, \dots, z^N, \bar{z}^1, \dots, \bar{z}^N)$ to $\Phi(z, \bar{z})$ for convenience.

In [22], it is shown that $e^{-\Phi/\hbar}$ corresponds to a vacuum projection operator $|\vec{0}\rangle \langle \vec{0}|$ for the noncommutative $\mathbb{C}P^N$. We extend this statement for general Kähler manifolds.

Proposition 3.3. *Let (M, ω) be a Kähler manifold, Φ be its Kähler potential with the property (3.14), and $*$ be a star product with separation of variables given in the previous section. Then the following function*

$$|\vec{0}\rangle \langle \vec{0}| := e^{-\Phi/\hbar}, \quad (3.16)$$

satisfies

$$\underline{a}_i * |\vec{0}\rangle \langle \vec{0}| = 0, \quad |\vec{0}\rangle \langle \vec{0}| * a_i^\dagger = 0, \quad (3.17)$$

$$\left(|\vec{0}\rangle \langle \vec{0}| \right) * \left(|\vec{0}\rangle \langle \vec{0}| \right) = e^{-\Phi/\hbar} * e^{-\Phi/\hbar} = e^{-\Phi/\hbar} = |\vec{0}\rangle \langle \vec{0}|. \quad (3.18)$$

Proof. We define the following normal ordered quantity,

$$: e^{-\sum_i a_i^\dagger \underline{a}_i} : := \prod_{i=1}^N \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)_*^n * (\underline{a}_i)_*^n. \quad (3.19)$$

Here $\prod_{i=1}^N$ is defined by $\prod_{i=1}^N f_i = f_1 * f_2 * \cdots * f_N$. Note that, if $i \neq j$, \underline{a}_i commutes with \underline{a}_j , a_j^\dagger , and a_i^\dagger commutes with \underline{a}_j , a_j^\dagger . Therefore $: e^{-\sum_i a_i^\dagger \underline{a}_i} :$ does not depend on the order of each factor $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)_*^n * (\underline{a}_i)_*^n$.

It is easy to show that $\underline{a}_i * \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)_*^n * (\underline{a}_i)_*^n = 0$, in the same way as in the case of the ordinary harmonic oscillator,

$$\underline{a}_i * \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)_*^n * (\underline{a}_i)_*^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[n (a_i^\dagger)_*^{n-1} (\underline{a}_i)_*^n + (a_i^\dagger)_*^n (\underline{a}_i)_*^{n+1} \right] = 0, \quad (3.20)$$

where the commutation relations (3.2) are used. Similarly, we can show $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)_*^n * (\underline{a}_i)_*^n * a_i^\dagger = 0$. These results and the fact that \underline{a}_i and a_i^\dagger commute with \underline{a}_j and a_j^\dagger for $i \neq j$ lead to $\underline{a}_i * : e^{-\sum_i a_i^\dagger \underline{a}_i} : = 0$ and $: e^{-\sum_i a_i^\dagger \underline{a}_i} : * a_i^\dagger = 0$. Further, these relations imply $: e^{-\sum_i a_i^\dagger \underline{a}_i} : * : e^{-\sum_i a_i^\dagger \underline{a}_i} : = : e^{-\sum_i a_i^\dagger \underline{a}_i} :$.

Therefore, all we have to do is to show

$$: e^{-\sum_i a_i^\dagger \underline{a}_i} := e^{-\Phi/\hbar}. \quad (3.21)$$

This can be done as follows:

$$\begin{aligned} : e^{-\sum_i a_i^\dagger \underline{a}_i} : &= \sum_{\vec{n}} \frac{(-1)^{|\vec{n}|}}{\vec{n}!} (a^\dagger)_{*}^{\vec{n}} * (\underline{a})_{*}^{\vec{n}} \\ &= \sum_{\vec{n}} \frac{(-1)^{|\vec{n}|}}{\vec{n}! \hbar^{|\vec{n}|}} (z)_{*}^{\vec{n}} * (\partial \Phi)_{*}^{\vec{n}}. \end{aligned} \quad (3.22)$$

In this paper, we use the following notation: for an N -tuple A_i ($i = 1, 2, \dots, N$) and an N -vector $\vec{n} = (n_1, n_2, \dots, n_N)$,

$$(A)_{*}^{\vec{n}} = (A_1)_{*}^{n_1} * (A_2)_{*}^{n_2} * \cdots * (A_N)_{*}^{n_N}, \quad (3.23)$$

$$\vec{n}! = n_1! n_2! \cdots n_N!, \quad |n| = \sum_{i=1}^N n_i. \quad (3.24)$$

By using $(z)_*^{\vec{n}} = (z)^{\vec{n}} = (z^1)^{n_1} \dots (z^N)^{n_N}$ and (2.25), (3.22) is recast as

$$\begin{aligned} & \sum_{n_1, n_2, \dots, n_N=0}^{\infty} \frac{1}{n_1! n_2! \dots n_N!} (-z^1)^{n_1} \dots (-z^N)^{n_N} e^{-\frac{\Phi(z, \bar{z})}{\hbar}} \partial_1^{n_1} \dots \partial_N^{n_N} e^{\frac{\Phi(z, \bar{z})}{\hbar}} \\ &= e^{-\frac{\Phi(z, \bar{z})}{\hbar}} e^{\frac{\Phi(0, \bar{z})}{\hbar}} \\ &= e^{-\frac{\Phi(z, \bar{z})}{\hbar}}. \end{aligned} \quad (3.25)$$

Here, the final equality follows from the condition (3.14). \square

From a similar calculation to the above proof, we can also show the following relations with respect to a_i and \underline{a}_i^\dagger ,

$$|\vec{0}\rangle\langle\vec{0}| = e^{-\Phi/\hbar} =: e^{-\sum_i \underline{a}_i^\dagger a_i} := \prod_{i=1}^N \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\underline{a}_i^\dagger)_*^n * (a_i)_*^n, \quad (3.26)$$

$$a_i * |\vec{0}\rangle\langle\vec{0}| = 0, \quad |\vec{0}\rangle\langle\vec{0}| * \underline{a}_i^\dagger = 0. \quad (3.27)$$

Lemma 3.4 (Sako, Suzuki, Umetsu [22]). $e^{-\Phi/\hbar} = |0\rangle\langle 0|$ satisfies the relation

$$|0\rangle\langle 0| * f(z, \bar{z}) = e^{-\Phi/\hbar} * f(z, \bar{z}) = e^{-\Phi/\hbar} f(0, \bar{z}) = |0\rangle\langle 0| f(0, \bar{z}), \quad (3.28)$$

$$f(z, \bar{z}) * |0\rangle\langle 0| = f(z, \bar{z}) * e^{-\Phi/\hbar} = f(z, 0) e^{-\Phi/\hbar} = f(z, 0) |0\rangle\langle 0|. \quad (3.29)$$

for a function $f(z, \bar{z})$ such that $f(z, \bar{w})$ can be expanded as Taylor series with respect to z^i and \bar{w}^j , respectively.

This proof is given in [22], but for the convenience its proof is reviewed here.

Proof. To show the relation (3.28), we note that the differential operator R_{z^i} corresponding to the right product of z^i contains only partial derivatives by \bar{z}^j , and thus commutes with z^k . Moreover, R_{z^i} annihilates $e^{-\Phi/\hbar}$, $R_{z^i} e^{-\Phi/\hbar} = e^{-\Phi/\hbar} * z^i = |\vec{0}\rangle\langle\vec{0}| * \underline{a}_i^\dagger = 0$. From these and (2.23), the relation (3.28) is shown as

$$\begin{aligned} e^{-\Phi/\hbar} * f(z, \bar{z}) &= R_f e^{-\Phi/\hbar} \\ &= \sum_{k_1, \dots, k_N=0}^{\infty} \frac{1}{k_1! \dots k_N!} \partial_1^{k_1} \dots \partial_N^{k_N} f(z, \bar{z}) \prod_{i=1}^N (R_{z^i} - z^i)^{k_i} e^{-\Phi/\hbar} \\ &= \sum_{k_1, \dots, k_N=0}^{\infty} \frac{1}{k_1! \dots k_N!} \partial_1^{k_1} \dots \partial_N^{k_N} f(z, \bar{z}) \prod_{i=1}^N (-z^i)^{k_i} e^{-\Phi/\hbar} \\ &= e^{-\Phi/\hbar} f(0, \bar{z}). \end{aligned} \quad (3.30)$$

Similarly, (3.29) follows from (2.19) and (3.27). \square

We expand a function $\exp \Phi(z, \bar{z})/\hbar$ as a power series,

$$e^{\Phi(z, \bar{z})/\hbar} = \sum_{\vec{m}, \vec{n}} H_{\vec{m}, \vec{n}}(z)^{\vec{m}}(\bar{z})^{\vec{n}} \quad (3.31)$$

where $(z)^{\vec{n}} = (z^1)^{n_1} \dots (z^N)^{n_N}$ and $(\bar{z})^{\vec{n}} = (\bar{z}^1)^{n_1} \dots (\bar{z}^N)^{n_N}$. Since $\exp \Phi/\hbar$ is real and satisfies (3.14), the expansion coefficients $H_{\vec{m}, \vec{n}}$ obey

$$\bar{H}_{\vec{m}, \vec{n}} = H_{\vec{n}, \vec{m}}, \quad (3.32)$$

$$H_{\vec{0}, \vec{n}} = H_{\vec{n}, \vec{0}} = \delta_{\vec{n}, \vec{0}}. \quad (3.33)$$

Using this expansion, the following relations are obtained.

Proposition 3.5. *The right $*$ -multiplication of $(\underline{a})_*^{\vec{n}} = (\partial\Phi/\hbar)_*^{\vec{n}}$ on $|\vec{0}\rangle\langle\vec{0}|$ is related to the right $*$ -multiplication of $(a)_*^{\vec{n}} = (\bar{z})_*^{\vec{n}}$ on $|\vec{0}\rangle\langle\vec{0}|$ as follows,*

$$\begin{aligned} |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_*^{\vec{n}} &= |\vec{0}\rangle\langle\vec{0}| * \left(\frac{1}{\hbar} \partial\Phi \right)_*^{\vec{n}} \\ &= \vec{n}! \sum_{\vec{m}} H_{\vec{n}, \vec{m}} |\vec{0}\rangle\langle\vec{0}| * (\bar{z})_*^{\vec{m}} = \vec{n}! \sum_{\vec{m}} H_{\vec{n}, \vec{m}} |\vec{0}\rangle\langle\vec{0}| * (a)_*^{\vec{m}}. \end{aligned} \quad (3.34)$$

Similarly, the following relation holds,

$$\begin{aligned} (\underline{a}^\dagger)_*^{\vec{n}} * |\vec{0}\rangle\langle\vec{0}| &= \left(\frac{1}{\hbar} \bar{\partial}\Phi \right)_*^{\vec{n}} * |\vec{0}\rangle\langle\vec{0}| \\ &= \vec{n}! \sum_{\vec{m}} H_{\vec{m}, \vec{n}}(z)^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| = \vec{n}! \sum_{\vec{m}} H_{\vec{m}, \vec{n}}(a^\dagger)_*^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}|. \end{aligned} \quad (3.35)$$

Proof. By using (2.24),

$$\begin{aligned} |\vec{0}\rangle\langle\vec{0}| * \left(\frac{1}{\hbar} \partial\Phi \right)_*^{\vec{n}} &= |\vec{0}\rangle\langle\vec{0}| * \left(e^{-\Phi/\hbar} (\partial)^{\vec{n}} e^{\Phi/\hbar} \right) \\ &= |\vec{0}\rangle\langle\vec{0}| * \left(e^{-\Phi/\hbar} \sum_{\vec{m}} \vec{n}! H_{\vec{n}, \vec{m}}(\bar{z})^{\vec{m}} + O(z) \right). \end{aligned} \quad (3.36)$$

From (3.28) and Lemma 3.2, this is rewritten as

$$\begin{aligned} |\vec{0}\rangle\langle\vec{0}| * \left(e^{-\Phi/\hbar} \sum_{\vec{m}} \vec{n}! H_{\vec{n}, \vec{m}}(\bar{z})^{\vec{m}} \right) \Big|_{(z, \bar{z})=(0, \bar{z})} &= |\vec{0}\rangle\langle\vec{0}| * \sum_{\vec{m}} \vec{n}! H_{\vec{n}, \vec{m}}(\bar{z})^{\vec{m}} \\ &= |\vec{0}\rangle\langle\vec{0}| * \sum_{\vec{m}} \vec{n}! H_{\vec{n}, \vec{m}}(a)_*^{\vec{m}}. \end{aligned} \quad (3.37)$$

The relation (3.35) is the complex conjugation of (3.34). \square

If there exists the inverse matrix $H_{\vec{m},\vec{n}}^{-1}$, then the following relations also holds,

Corollary 3.6.

$$|\vec{0}\rangle\langle\vec{0}| * (a)_{*}^{\vec{n}} = \sum_{\vec{m}} \frac{1}{\vec{m}!} H_{\vec{n},\vec{m}}^{-1} |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_{*}^{\vec{m}}, \quad (3.38)$$

$$(a^{\dagger})_{*}^{\vec{n}} * |\vec{0}\rangle\langle\vec{0}| = \sum_{\vec{m}} \frac{1}{\vec{m}!} H_{\vec{m},\vec{n}}^{-1} (\underline{a}^{\dagger})^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}|, \quad (3.39)$$

where $H_{\vec{n},\vec{m}}^{-1}$ is the inverse matrix of the matrix $H_{\vec{n},\vec{m}}$, $\sum_{\vec{k}} H_{\vec{n},\vec{k}} H_{\vec{k},\vec{n}}^{-1} = \delta_{\vec{n},\vec{n}}$.

We introduce bases of the Fock representation as follows,

$$|\vec{m}\rangle\langle\vec{n}| := \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (a^{\dagger})_{*}^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_{*}^{\vec{n}} = \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (z)_{*}^{\vec{m}} * e^{-\Phi/\hbar} * \left(\frac{1}{\hbar} \partial\Phi \right)_{*}^{\vec{n}}. \quad (3.40)$$

By using (3.34), the bases are also written as

$$\begin{aligned} |\vec{m}\rangle\langle\vec{n}| &= \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{n},\vec{k}} (z)_{*}^{\vec{m}} * e^{-\Phi/\hbar} * (\bar{z})_{*}^{\vec{k}} \\ &= \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{n},\vec{k}} (z)_{*}^{\vec{m}} (\bar{z})^{\vec{k}} e^{-\Phi/\hbar}. \end{aligned} \quad (3.41)$$

The completeness of the bases are formally shown as

$$\begin{aligned} \sum_{\vec{n}} |\vec{n}\rangle\langle\vec{n}| &= \sum_{\vec{m},\vec{n}} H_{\vec{n},\vec{m}} (z)_{*}^{\vec{n}} (\bar{z})^{\vec{m}} e^{-\Phi/\hbar} \\ &= e^{\Phi/\hbar} e^{-\Phi/\hbar} \\ &= 1. \end{aligned} \quad (3.42)$$

The $*$ -products between the bases are calculated as

$$\begin{aligned} |\vec{m}\rangle\langle\vec{n}| * |\vec{k}\rangle\langle\vec{l}| &= \frac{1}{\sqrt{\vec{m}!\vec{n}!\vec{k}!\vec{l}!}} (a^{\dagger})_{*}^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_{*}^{\vec{n}} * (a^{\dagger})_{*}^{\vec{k}} * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_{*}^{\vec{l}} \\ &= \delta_{\vec{n},\vec{k}} |\vec{m}\rangle\langle\vec{l}|. \end{aligned} \quad (3.43)$$

The behavior of the bases under the complex conjugation is different from usual,

$$\begin{aligned} \overline{|\vec{m}\rangle\langle\vec{n}|} &= \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{k},\vec{n}} (z)^{\vec{k}} (\bar{z})^{\vec{m}} e^{-\Phi/\hbar} \\ &= \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k},\vec{l}} \sqrt{\frac{\vec{k}!}{\vec{l}!}} H_{\vec{k},\vec{n}} H_{\vec{m},\vec{l}}^{-1} |\vec{k}\rangle\langle\vec{l}|. \end{aligned} \quad (3.44)$$

The creation and annihilation operators $a_i^\dagger, \underline{a}_i$ act on the bases as follows,

$$a_i^\dagger * |\vec{m}\rangle \langle \vec{n}| = \sqrt{m_i + 1} |\vec{m} + \vec{e}_i\rangle \langle \vec{n}|, \quad (3.45)$$

$$\underline{a}_i * |\vec{m}\rangle \langle \vec{n}| = \sqrt{m_i} |\vec{m} - \vec{e}_i\rangle \langle \vec{n}|, \quad (3.46)$$

$$|\vec{m}\rangle \langle \vec{n}| * a_i^\dagger = \sqrt{n_i} |\vec{m}\rangle \langle \vec{n} - \vec{e}_i|, \quad (3.47)$$

$$|\vec{m}\rangle \langle \vec{n}| * \underline{a}_i = \sqrt{n_i + 1} |\vec{m}\rangle \langle \vec{n} + \vec{e}_i|, \quad (3.48)$$

where \vec{e}_i is a unit vector, $(\vec{e}_i)_j = \delta_{ij}$. The action of a_i and \underline{a}_i^\dagger is derived by the Hermitian conjugation of the above equations.

The creation and annihilation operators can be expanded with respect to the bases,

$$a_i^\dagger = \sum_{\vec{n}} \sqrt{n_i + 1} |\vec{n} + \vec{e}_i\rangle \langle \vec{n}|, \quad (3.49)$$

$$\underline{a}_i = \sum_{\vec{n}} \sqrt{n_i + 1} |\vec{n}\rangle \langle \vec{n} + \vec{e}_i|, \quad (3.50)$$

$$a_i = \sum_{\vec{m}, \vec{n}, \vec{k}} \sqrt{\frac{\vec{m}!}{\vec{n}!}} H_{\vec{m}, \vec{k}} H_{\vec{k} + \vec{e}_i, \vec{n}}^{-1} |\vec{m}\rangle \langle \vec{n}|, \quad (3.51)$$

$$\underline{a}_i^\dagger = \sqrt{\frac{\vec{m}!}{\vec{n}!}} (k_i + 1) H_{\vec{m}, \vec{k} + \vec{e}_i} H_{\vec{k}, \vec{n}}^{-1} |\vec{m}\rangle \langle \vec{n}|. \quad (3.52)$$

4 Transition maps

Let $\{U_a\}$ with $M = \cup_a U_a$ be a locally finite open covering and $\{(U_a, \phi_a)\}$ be an atlas, where $\phi_a : U_a \rightarrow \mathbb{C}^N$. Consider the case $U_a \cap U_b \neq \emptyset$. Denote by $\phi_{a,b}$ the transition map from $\phi_a(U_a)$ to $\phi_b(U_b)$. The local coordinates $(z, \bar{z}) = (z^1, \dots, z^N, \bar{z}^1, \dots, \bar{z}^N)$ on U_a are transformed into the coordinates $(w, \bar{w}) = (w^1, \dots, w^N, \bar{w}^1, \dots, \bar{w}^N)$ on U_b by $(w, \bar{w}) = (w(z), \bar{w}(\bar{z}))$, where $w(z) = (w^1(z), \dots, w^N(z))$ is a holomorphic function and $\bar{w}(\bar{z}) = (\bar{w}^1(\bar{z}), \dots, \bar{w}^N(\bar{z}))$ is an anti-holomorphic function. Denote by $f *_a g$ and $f *_b g$ the star products defined in Section 2 on U_a and U_b , respectively. In general, there is a nontrivial transition maps T between two star products i.e. $f *_b g = T(f) *_a T(g)$. But the transition maps are trivial in our case.

Proposition 4.1. *For an overlap $U_a \cap U_b \neq \emptyset$,*

$$f *_b g(w, \bar{w}) = \phi_{a,b}^* f *_a g(w, \bar{w}) = \phi_{a,b}^* f(w(z), \bar{w}(\bar{z})) *_a g(w(z), \bar{w}(\bar{z})). \quad (4.1)$$

Here $\phi_{a,b}^*$ is the pull back of $\phi_{a,b}$.

Proof. The Kähler potentials $\Phi_a(z, \bar{z})$ on U_a and $\Phi_b(w, \bar{w})$ on U_b satisfy, in general,

$$\Phi_b(w, \bar{w}) = \Phi_a(z, \bar{z}) + \phi(z) + \bar{\phi}(\bar{z}),$$

where ϕ is a holomorphic function and $\bar{\phi}$ is an anti-holomorphic function. We define a differential operator $L_{b,f}$ by $L_{b,f}g := f *_b g$ on U_b . Similarly, we use $g_b^{\bar{i}j}$, $D_b^{\bar{i}}$ etc. as the metric on U_b , differential operator $D_b^{\bar{i}}$ on U_b , etc. As mentioned in Section 2,

$$L_{b,f} = \sum_{n=0}^{\infty} \hbar^n a_{n,\bar{i}}^b(f) D_b^{\bar{i}} = \sum_{n=0}^{\infty} \hbar^n \sum_{k \geq 0} a_{\bar{i}_1 \dots \bar{i}_k}^{b(n;k)} D_b^{\bar{i}_1} \dots D_b^{\bar{i}_k}, \quad (4.2)$$

is determined by

$$[L_{b,f}, R_{b,\partial_l \Phi_b}] = [L_{b,f}, \frac{\partial \Phi_b}{\partial \bar{w}^l} + \hbar \frac{\partial}{\partial \bar{w}^l}] = 0 \quad (4.3)$$

On the overlap $U_a \cap U_b$,

$$D_b^{\bar{i}} = \frac{\partial \bar{w}^i}{\partial \bar{z}^l} D_a^{\bar{l}}, \quad (4.4)$$

because $g_b^{\bar{i}j} = \frac{\partial \bar{w}^i}{\partial \bar{z}^k} \frac{\partial w^j}{\partial z^l} g_a^{\bar{k}l}$. From the fact that differential operators $D_b^{\bar{i}}$ contain only differentiation with respect to holomorphic coordinates w^i , $D_b^{\bar{i}}$ commutes with anti-holomorphic functions, then we obtain

$$L_{b,f} = \sum_{n=0}^{\infty} \hbar^n a_{n,\bar{i}}^b(f) \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)^{\bar{i}}_{\bar{j}} D_b^{\bar{j}}, \quad (4.5)$$

where $\left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)^{\bar{i}}_{\bar{j}}$ is an anti-holomorphic function

$$\left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)^{\bar{i}}_{\bar{j}} = \frac{\partial \bar{w}^{i_1}}{\partial \bar{z}^{j_1}} \dots \frac{\partial \bar{w}^{i_k}}{\partial \bar{z}^{j_k}}. \quad (4.6)$$

Here, the Einstein summation convention over repeated indices is also used for multi indices like \bar{i} and so on. Then

$$\begin{aligned} & \left[L_{b,f}, \frac{\partial \Phi_b}{\partial \bar{w}^l} + \hbar \frac{\partial}{\partial \bar{w}^l} \right] \\ &= \left[\sum_{n=0}^{\infty} \hbar^n a_{n,\bar{i}}^b(f) \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)^{\bar{i}}_{\bar{j}} D_b^{\bar{j}}, \frac{\partial \bar{z}^k}{\partial \bar{w}^l} \left(\frac{\partial \Phi_a}{\partial \bar{z}^k} + \frac{\partial \bar{\phi}}{\partial \bar{z}^k} + \hbar \frac{\partial}{\partial \bar{z}^k} \right) \right] \\ &= \frac{\partial \bar{z}^k}{\partial \bar{w}^l} \left[\sum_{n=0}^{\infty} \hbar^n a_{n,\bar{i}}^b(f) \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)^{\bar{i}}_{\bar{j}} D_b^{\bar{j}}, \frac{\partial \Phi_a}{\partial \bar{z}^k} + \hbar \frac{\partial}{\partial \bar{z}^k} \right] = 0, \end{aligned} \quad (4.7)$$

and thus we obtain

$$L_{a,f} = \sum_{n=0}^{\infty} \hbar^n a_{n,\vec{i}}^b(f) \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)_{\vec{j}}^{\vec{i}} D_a^{\vec{j}} = L_{b,f} \quad (4.8)$$

which satisfies the condition $[L_{a,f}, R_{a,\partial_{\vec{i}}\Phi_a}] = 0$. \square

(4.8) means that

$$a_{n,\vec{j}}^b(f) \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)_{\vec{i}}^{\vec{j}} = a_{n,\vec{i}}^a(f), \quad (4.9)$$

in other words, $a_{n,\alpha}^b(f)$ transforms as a tensor.

As a next step, we consider the transition function between twisted Fock representations. From Lemma 3.2, we can choose $\Phi_a(z, \bar{z})$ and $\Phi_b(w, \bar{w})$ such that

$$\Phi_a(0, \bar{z}) = \Phi_a(z, 0) = 0, \quad \Phi_b(0, \bar{w}) = \Phi_b(w, 0) = 0. \quad (4.10)$$

Using these Kähler potentials, $|\vec{0}\rangle_{pp}\langle\vec{0}|$ is defined as

$$|\vec{0}\rangle_{pp}\langle\vec{0}| = e^{-\Phi_p/\hbar}, \quad (p = a, b),$$

and $|\vec{m}\rangle_{pp}\langle\vec{n}|$ are defined by

$$\begin{aligned} |\vec{m}\rangle_{aa}\langle\vec{n}| &= \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (z)_{*}^{\vec{m}} * e^{-\Phi_a/\hbar} * \left(\frac{1}{\hbar} \partial \Phi_a \right)_{*}^{\vec{n}}, \\ |\vec{m}\rangle_{bb}\langle\vec{n}| &= \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (w)_{*}^{\vec{m}} * e^{-\Phi_b/\hbar} * \left(\frac{1}{\hbar} \partial \Phi_b \right)_{*}^{\vec{n}}. \end{aligned}$$

Let us consider the case that on the overlap $U_a \cap U_b$ the coordinate transition function $w(z)$, $\bar{w}(\bar{z})$, and the functions $\exp(\phi(w)/\hbar)$ and $\exp(\bar{\phi}(\bar{w})/\hbar)$ are given by analytic functions. Then the products $(w(z))^{\vec{\alpha}} \exp -(\phi(w)/\hbar)$ and $(\bar{w}(\bar{z}))^{\vec{\alpha}} \exp -(\bar{\phi}(\bar{w})/\hbar)$ are also analytic functions;

$$\begin{aligned} (w(z))^{\vec{\alpha}} e^{-\phi(w)/\hbar} &= \sum_{\vec{\beta}} C_{\vec{\beta}}^{\vec{\alpha}} z^{\vec{\beta}}, \\ (\bar{w}(\bar{z}))^{\vec{\alpha}} e^{-\bar{\phi}(\bar{w})/\hbar} &= \sum_{\vec{\beta}} \bar{C}_{\vec{\beta}}^{\vec{\alpha}} \bar{z}^{\vec{\beta}}. \end{aligned} \quad (4.11)$$

By using (3.34), the bases are also written as

$$\begin{aligned} |\vec{m}\rangle_{aa}\langle\vec{n}| &= \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{n},\vec{k}}^a(z) \vec{m}(\bar{z})^{\vec{k}} e^{-\Phi_a/\hbar}, \\ |\vec{m}\rangle_{bb}\langle\vec{n}| &= \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{n},\vec{k}}^b(w) \vec{m}(\bar{w})^{\vec{k}} e^{-\Phi_b/\hbar}. \end{aligned} \quad (4.12)$$

From the (4.11)

$$|\vec{m}\rangle_{bb}\langle\vec{n}| = \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{n},\vec{k}}^b \left(\sum_{\vec{\alpha}} C_{\vec{\alpha}}^{\vec{m}} z^{\vec{\alpha}} \right) \left(\sum_{\vec{\beta}} \bar{C}_{\vec{\beta}}^{\vec{k}} \bar{z}^{\vec{\beta}} \right) e^{-\Phi_a/\hbar}. \quad (4.13)$$

Finally, we obtain transformation between the bases,

$$T^{ab} : F_{U_a} \rightarrow F_{U_b}, \quad (4.14)$$

as

$$|\vec{m}\rangle_{bb}\langle\vec{n}| = \sum_{\vec{i},\vec{j}} T_{\vec{m}\vec{n}}^{ba,\vec{i}\vec{j}} |\vec{i}\rangle_{aa}\langle\vec{j}|, \quad (4.15)$$

where

$$T_{\vec{m}\vec{n}}^{ba,\vec{i}\vec{j}} = \sqrt{\frac{\vec{n}!}{\vec{m}!}} \sqrt{\frac{\vec{i}!}{\vec{j}!}} \sum_{\vec{k}} H_{\vec{n},\vec{k}}^b (C_{\vec{i}}^{\vec{m}}) \left(\sum_{\vec{\beta}} \bar{C}_{\vec{\beta}}^{\vec{k}} H_{\vec{\beta}\vec{j}}^{a-1} \right). \quad (4.16)$$

Using this transformation, the twisted Fock representation is extended to M . We call it the twisted Fock representation of M .

5 Trace operation

A trace operation to the Fock algebra is studied in this section. A trace density μ of noncommutative manifolds $(M, *)$ is defined as a density such that

$$\int_M \mu f * g = \int_M \mu g * f \quad (5.1)$$

for any functions $f \in C^\infty(M)$ and $g \in C_0^\infty(M)$, where C_0^∞ is used as a set of compactly supported smooth functions. Let $\text{Tr}_M : C^\infty(M) \rightarrow \mathbb{C} \cup \infty$ be an integration with this trace density μ :

$$\text{Tr}_M f := \int_M \mu f. \quad (5.2)$$

The existence of the trace density for the noncommutative Kähler manifolds with the deformation quantization with separation of variables are guaranteed by the study in [13]. Note that elements of the basis of the twisted Fock algebras $|\vec{n}\rangle\langle\vec{m}|$ are not necessarily compactly supported functions on an each local coordinate open set, in general.

Let $\{(U_p, \phi_p)\}$ is an atlas of a Kähler manifold M and we use the notation $V_p = \phi_p(U_p) \subset \mathbb{R}^{2N}$. For a bounded function given in the form of $f * g$, an integral over $M = \cup_p U_p$ of $f * g$ with the trace density μ_p on V_p is defined such as

$$\int_M f * g \mu = \sum_p \int_{V_p} \rho_p f * g \mu_p, \quad (5.3)$$

where ρ_p is a partition of unity. Note that

$$\sum_p \int_{V_p} \rho_p (f * g) \mu_p = \sum_p \int_{V_p} (\rho_p f) * g \mu_p, \quad (5.4)$$

because ρ_p is an element of a partition of unity. Therefore for a compact closed Kähler manifold M , cyclic symmetries (5.1) hold for arbitrary $f, g \in C^\infty(M)$. In the following of this section, we fix a partition of unity on M .

Let us define a linear operation for the Fock algebra as follows.

Definition 4. Let (U_p, ϕ_p) be a chart of M . The local linear operation Sp_p on each U_p is defined as a linear map from F_{U_p} to $\mathbb{C} \cup \infty$ such that

$$\text{Sp}_p A * B = \text{Sp}_p B * A \quad (5.5)$$

for arbitrary elements A and B of twisted Fock representation, and

$$\text{Sp}_p |\vec{0}\rangle_{pp} \langle \vec{0}| = c_p. \quad (5.6)$$

Here c_p is a constant depending on p .

Note that for the case that the considering Kähler manifold M is \mathbb{C}^n , the $\text{Sp}_{\mathbb{C}^n}$ is equal to the trace operation Tr up to the c_p . (See Example 1. in Section 6.)

Remark. The cyclic symmetry of the Sp_p operation and the commutation relations of the creation and annihilation operators determine the the Sp_p of bases of the twisted Fock representation.

$$\text{Sp}_p |\vec{n}\rangle_{pp} \langle \vec{m}| = c_p \delta_{\vec{n}\vec{m}} = c_p \delta_{n_1 m_1} \delta_{n_2 m_2} \cdots \delta_{n_N m_N}. \quad (5.7)$$

Indeed, the fact that the trace of commutator $|\vec{n}\rangle_{pp}\langle\vec{m}|$ and number operator $N_i = a_i^\dagger \underline{a}_i$ is zero,

$$0 = \text{Sp}_p[N_i, |\vec{n}\rangle_{pp}\langle\vec{m}|]_* = (n_i - m_i) \text{Tr}_p |\vec{n}\rangle_{pp}\langle\vec{m}|, \quad (i = 1, \dots, N), \quad (5.8)$$

implies

$$\text{Sp}_p |\vec{n}\rangle_{pp}\langle\vec{m}| = c \delta_{\vec{n}\vec{m}}, \quad (5.9)$$

where c is some constant. Furthermore, because of $|\vec{n}\rangle_p = \frac{1}{\sqrt{n_i}} a_i^\dagger |\vec{n} - \vec{e}_i\rangle_p$,

$$\begin{aligned} \text{Sp}_p |\vec{n}\rangle_{pp}\langle\vec{n}| &= \frac{1}{n_i} \text{Sp}_p a_i^\dagger |\vec{n} - \vec{e}_i\rangle_{pp}\langle\vec{n} - \vec{e}_i|_{\underline{a}_i} \\ &= \frac{1}{n_i} \text{Sp}_p (N_i + 1) |\vec{n} - \vec{e}_i\rangle_{pp}\langle\vec{n} - \vec{e}_i| \\ &= \text{Sp}_p |\vec{n} - \vec{e}_i\rangle_{pp}\langle\vec{n} - \vec{e}_i| \end{aligned} \quad (5.10)$$

$$\begin{aligned} &\vdots \\ &= \text{Sp}_p |\vec{0}\rangle_{pp}\langle\vec{0}| = c_p. \end{aligned} \quad (5.11)$$

These results show the equation (5.7). Note that (5.7) are derived by using only commutation relations between the a_i^\dagger 's and \underline{a}_i 's and cyclic symmetry $\text{Sp}_p A * B = \text{Sp}_p B * A$.

Let us consider the relation between the trace operations and integration.

Proposition 5.1. *If cyclic symmetries*

$$\int_U \mu_U (a^\dagger)_*^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_*^{\vec{n}} = \int_U \mu_U |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_*^{\vec{n}} * (a^\dagger)_*^{\vec{m}} \quad (5.12)$$

are satisfied for arbitrary \vec{n}, \vec{m} , and $c_0 := \int_U \mu_U |\vec{0}\rangle\langle\vec{0}| = \int_U \mu_U \exp(-\Phi/\hbar)$ is finite, then

$$\int_U \mu_U |\vec{m}\rangle\langle\vec{n}| = c_0 \delta_{\vec{m}, \vec{n}}. \quad (5.13)$$

Proof.

$$\begin{aligned} \int_U \mu_U |\vec{m}\rangle\langle\vec{n}| &= \frac{1}{\sqrt{\vec{m}!\vec{n}!}} \int_U \mu_U (a^\dagger)_*^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_*^{\vec{n}} \\ &= \delta_{\vec{m}, \vec{n}} \int_U \mu_U |\vec{0}\rangle\langle\vec{0}| \\ &= c_0 \delta_{\vec{m}, \vec{n}}. \end{aligned} \quad (5.14)$$

□

For example, for \mathbb{C}^N and $\mathbb{C}H^N$ we can chose U and Φ to satisfy the conditions in Proposition 5.1. In such cases, Sp operation on U

$$\text{Sp}_U |\vec{m}\rangle \langle \vec{n}| := \delta_{\vec{m}\vec{n}} c_U \quad (5.15)$$

is expressed by the above integration, i.e.

$$\text{Sp}_U |\vec{m}\rangle \langle \vec{n}| = \frac{c_U}{c_0} \int_U \mu_g |\vec{m}\rangle \langle \vec{n}|. \quad (5.16)$$

Here c_U is a some constant. Then the results of the trace operation of the twisted Fock algebra are given by easy algebraic calculation of Sp.

For general Kähler manifolds there might not exist the open covering $M = \cup_p U_p$ such that each U_p satisfies conditions (5.12) in this proposition. In such case, (5.16) does not work. In addition, we have to introduce a partition of unity to describe the integration over a whole manifold, in general case. But it is unknown whether the partition of unity belongs to the twisted Fock algebra or not for the general case. Therefore, we can not naively compare Sp operations with Tr. If globally defined twisted Fock algebra exists on a Kähler manifolds, then integral is evaluated by algebraic process. This problem is discussed for some cases in Section 6.

Then how can we estimate the trace operation by using Sp? The Sp is related with integral over V_p under some conditions.

Proposition 5.2. *Let (U_p, ϕ_p) be a chart satisfying $\phi_p(U_p) = V_p \subset \mathbb{R}^{2N}$ and ρ_p be a partition of unity corresponding to U_p . Consider that volumes of every overlapping domains between U_p and U_q ($q \neq p$) are bounded by arbitrary positive number ϵ , and values of all commutators between ρ_p and a_i^\dagger or \underline{a}_j are bounded by $1/\epsilon^{1-\delta_p}$ where δ_p is a real number. Then the integration is related with Sp*

$$\text{Sp}_p |\vec{n}\rangle_{pp} \langle \vec{m}| = \int_{V_p} \rho_p |\vec{n}\rangle_{pp} \langle \vec{m}| \mu_g + O(\epsilon^{\delta_p}), \quad (5.17)$$

with the c_p is a constant given by

$$c_p = \int_{V_p} e^{-\Phi_p/\hbar} \mu_g. \quad (5.18)$$

Proof. At the above remark, we made sure that this Sp_p operation is determined by the cyclic symmetry of the trace and the algebraic relations of a_i^\dagger 's and \underline{a}_i 's. By the partition of unity ρ_p , the operation $\int_{V_p} \mu_g$ has cyclic symmetry for any elements in the twisted Fock algebra. The only problem is ρ_p does not commute with a_i^\dagger 's and \underline{a}_i 's on the over lapping region. From the condition that volumes of every overlapping domains between U_p and U_q ($q \neq p$) are bounded by an arbitrary positive number ϵ , and values of all commutators between ρ_p and a_i^\dagger or \underline{a}_j are bounded by $1/\epsilon^{1-\delta}$, (5.17) is trivial. \square

6 Examples

In this section, some examples of the Fock representations are given.

Example 1: Fock representation of \mathbb{C}^N

The first example is \mathbb{C}^N . The Kähler potential is given by

$$\Phi_{\mathbb{C}^N} = \sum_{i=1}^N |z^i|^2. \quad (6.1)$$

By the process given in Section 2, the star product is easily obtained as

$$f * g = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \delta^{k_1 l_1} \dots \delta^{k_n l_n} (\partial_{\bar{k}_1} \dots \partial_{\bar{k}_n} f) (\partial_{l_1} \dots \partial_{l_n} g).$$

This star product was given in [12]. We put

$$a_i^\dagger = z^i, \quad \underline{a}_i = \frac{1}{\hbar} \bar{z}^i, \quad a_i = \bar{z}^i, \quad \underline{a}_i^\dagger = \frac{1}{\hbar} z^i. \quad (6.2)$$

Then they satisfy the commutation relations:

$$[\underline{a}_i, a_j^\dagger]_* = \delta_{ij}, \quad [a_i, \underline{a}_j^\dagger]_* = \hbar \delta_{ij} \quad (6.3)$$

and the others are zero. Since in this case the operators with the underline are essentially equal to those without the underline, we omit the underline of the bra vectors. The basis of the twisted Fock algebra is given by

$$|\vec{m}\rangle \langle \vec{n}| = \frac{1}{\hbar^{|\vec{n}|} \sqrt{\vec{m}! \vec{n}!}} (z)^{\vec{m}} e^{-\Phi/\hbar} (\bar{z})^{\vec{n}}. \quad (6.4)$$

These are defined globally, so the trace operations for the twisted Fock algebras given by integral over \mathbb{C}^N is equal to the Sp operation with $c_{\mathbb{C}^N} = 1$:

$$\text{Tr}_{\mathbb{C}^N} |\vec{m}\rangle \langle \vec{n}| := \frac{\hbar^N}{\pi^N} \int_{\mathbb{C}^N} dx^{2D} |\vec{m}\rangle \langle \vec{n}| = \text{Sp}_{\mathbb{C}^N} |\vec{m}\rangle \langle \vec{n}| = \delta_{\vec{m}\vec{n}}. \quad (6.5)$$

These results coincide with well known facts for noncommutative Euclidean spaces.

Example 2 : Fock representation of a cylinder

The second example is a cylinder C . Let us consider the cylinder as a special case of \mathbb{C} with an equivalence relation $z \sim z + 2\pi$. We choose its open covering as $C = U_a \cup U_b$, where $U_a = \{z_a \in \mathbb{C} \mid -\frac{\pi}{2} < \text{Re } z_a < \pi\}$ and $U_b =$

$\{z_b \in \mathbb{C} | \frac{\pi}{2} < \text{Re } z_b < 2\pi\}$. Then there are two overlap regions. The first one is $A = \{z \in \mathbb{C} | \frac{\pi}{2} < \text{Re } z < \pi\} \subset U_a \cap U_b$, and the transition functions on it is given by the identity $z_a = z_b$. The other overlap is $B = \{z \in \mathbb{C} | \frac{3\pi}{2} < \text{Re } z < 2\pi\} \subset U_b$. On B , the transition function is given by $z_b = z_a + 2\pi$. The Kähler potential is defined by (6.1) for $N = 1$ and the star products on the cylinder is also given by (6.2) on each of the open subset U_a and U_b . The basis of the local twisted Fock algebra $|\vec{m}\rangle\langle\vec{n}|$ on U_a and U_b are also given as (6.4). However, we can not describe them globally since they do not have translation invariance under $z \rightarrow z + 2\pi$. Thus Tr_C can not be represented by using Sp_a and Sp_b .

Example 3 : Fock representation of noncommutative $\mathbb{C}P^N$

We give an explicit expression of the twisted Fock representation of noncommutative of $\mathbb{C}P^N$. In this case, the twisted Fock representation on an open set is essentially the same as the representation given in [22, 23, 16, 24]. (In a context of a Fuzzy $\mathbb{C}P^N$, which is a different approach to noncommutative $\mathbb{C}P^N$, the Fock representations are discussed in [1, 2, 8].)

Let denote ζ^a ($a = 0, 1, \dots, N$) homogeneous coordinates and $\bigcup U_a$ ($U_a = \{[\zeta^0 : \zeta^1 : \dots : \zeta^N] | \zeta^a \neq 0\}$) an open covering of $\mathbb{C}P^N$. We define inhomogeneous coordinates on U_a as

$$z_a^0 = \frac{\zeta^0}{\zeta^a}, \quad \dots, \quad z_a^{a-1} = \frac{\zeta^{a-1}}{\zeta^a}, \quad z_a^{a+1} = \frac{\zeta^{a+1}}{\zeta^a}, \quad \dots, \quad z_a^N = \frac{\zeta^N}{\zeta^a}. \quad (6.6)$$

We choose a Kähler potential on U_a which satisfies the condition (3.14)

$$\Phi_a = \ln(1 + |z_a|^2), \quad (6.7)$$

where $|z_a|^2 = \sum_i |z_a^i|^2$. A star product on U_a is given as follows [22, 23]:

$$f * g = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1 \bar{k}_1} \dots g_{j_n \bar{k}_n} (D^{j_1} \dots D^{j_n} f) D^{\bar{k}_1} \dots D^{\bar{k}_n} g, \quad (6.8)$$

where

$$c_n(\hbar) = \frac{\Gamma(1 - n + 1/\hbar)}{n! \Gamma(1 + 1/\hbar)}, \quad D^{\bar{i}} = g^{\bar{i}j} \partial_j, \quad D^i = g^{i\bar{j}} \partial_{\bar{j}}. \quad (6.9)$$

On U_a , creation and annihilation operators are given as

$$a_{a,i}^\dagger = z_a^i, \quad \underline{a_{a,i}} = \frac{1}{\hbar} \partial_i \Phi_a = \frac{\bar{z}_a^i}{\hbar(1 + |z_a|^2)}, \quad a_{a,i} = \bar{z}_a^i, \quad \underline{a_{a,i}}^\dagger = \frac{1}{\hbar} \partial_{\bar{i}} \Phi_a = \frac{z_a^i}{\hbar(1 + |z_a|^2)}. \quad (6.10)$$

and a vacuum is

$$|\vec{0}\rangle_{aa}\langle\vec{0}| = e^{-\Phi_a/\hbar} = (1 + |z_a|^2)^{-1/\hbar}. \quad (6.11)$$

Bases of the Fock representation on U_a are constructed as

$$\begin{aligned} |\vec{m}\rangle_{aa}\langle\vec{n}| &= \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (a_a^\dagger)_*^{\vec{m}} * |\vec{0}\rangle_{aa}\langle\vec{0}| * (a_a)_*^{\vec{n}} \\ &= \frac{1}{\sqrt{\vec{m}!\vec{n}!\hbar^{|\vec{n}|}}} (z_a)_*^{\vec{m}} * e^{-\Phi_a/\hbar} * (\partial\Phi_a)_*^{\vec{n}}. \end{aligned} \quad (6.12)$$

By using (3.28), (3.29) and the following relation which is shown in [22],

$$\begin{aligned} (\partial\Phi_a)_*^{\vec{n}} &= \frac{\hbar^{|\vec{n}|}\Gamma(1/\hbar + 1)}{\Gamma(1/\hbar - |\vec{n}| + 1)} (\partial\Phi_a)^{\vec{n}} \\ &= \frac{\hbar^{|\vec{n}|}\Gamma(1/\hbar + 1)}{\Gamma(1/\hbar - |\vec{n}| + 1)} \left(\frac{\bar{z}_a}{1 + |z_a|^2} \right)^{\vec{n}}, \end{aligned} \quad (6.13)$$

the bases can be explicitly written as

$$|\vec{m}\rangle_{aa}\langle\vec{n}| = \frac{\Gamma(1/\hbar + 1)}{\sqrt{\vec{m}!\vec{n}!\Gamma(1/\hbar - |\vec{n}| + 1)}} (z_a)^{\vec{m}} (\bar{z}_a)^{\vec{n}} e^{-\Phi/\hbar}. \quad (6.14)$$

By comparing this equation and (3.41), $H_{\vec{m},\vec{n}}$ is obtained as

$$H_{\vec{m},\vec{n}} = \delta_{\vec{m},\vec{n}} \frac{\Gamma(1/\hbar + 1)}{\vec{m}!\Gamma(1/\hbar - |\vec{m}| + 1)}, \quad (6.15)$$

and it is easily seen that this formally satisfies $e^{\Phi_a/\hbar} = \sum H_{\vec{m},\vec{n}}(z_a)^{\vec{m}}(\bar{z})^{\vec{n}}$.

Let us consider transformations between the Fock representations on U_a and U_b ($a < b$). The transformations for the coordinates and the Kähler potential on $U_a \cap U_b$ are

$$z_a^i = \frac{z_b^i}{z_b^a}, \quad (i = 0, 1, \dots, a-1, a+1, \dots, b-1, b+1, \dots, N), \quad z_a^b = \frac{1}{z_b^a}, \quad (6.16)$$

$$\Phi_a = \Phi_b - \ln z_b^a - \ln \bar{z}_b^a. \quad (6.17)$$

Thus, $|\vec{m}\rangle_{aa}\langle\vec{n}|$ is written on $U_a \cap U_b$ as

$$\begin{aligned} |\vec{m}\rangle_{aa}\langle\vec{n}| &= \frac{\Gamma(1/\hbar + 1)}{\sqrt{\vec{m}!\vec{n}!\Gamma(1/\hbar - |\vec{n}| + 1)}} e^{-\Phi_b/\hbar} \\ &\quad \times (z_b^0)^{m_0} \dots (z_b^{a-1})^{m_{a-1}} (z_b^a)^{1/\hbar - |\vec{m}|} (z_b^{a+1})^{m_{a+1}} \dots (z_b^{b-1})^{m_{b-1}} (z_b^{b+1})^{m_{b+1}} \dots (z_b^N)^{m_N} \\ &\quad \times (\bar{z}_b^0)^{n_0} \dots (\bar{z}_b^{a-1})^{n_{a-1}} (\bar{z}_b^a)^{1/\hbar - |\vec{n}|} (\bar{z}_b^{a+1})^{n_{a+1}} \dots (\bar{z}_b^{b-1})^{n_{b-1}} (\bar{z}_b^{b+1})^{n_{b+1}} \dots (\bar{z}_b^N)^{n_N}, \end{aligned} \quad (6.18)$$

where

$$\vec{m} = (m_0, \dots, m_{a-1}, m_{a+1}, \dots, m_N), \quad (6.19)$$

$$\vec{n} = (n_0, \dots, n_{a-1}, n_{a+1}, \dots, n_N). \quad (6.20)$$

We should treat $(z_b^a)^{1/\hbar-|m|}$ and $(\bar{z}_b^a)^{1/\hbar-|n|}$ carefully, because if they are not monomials some trick is needed to express them as the twisted Fock representation. We here make comments about the trick briefly. From the expression of the basis (6.14), a function $f(z, \bar{z})e^{-\Phi/\hbar}$ is expressed as the Twisted Fock algebra when $f(z, \bar{z})$ is given as a Taylor expansion in z and \bar{z} . For simplicity, we consider the one dimensional case. When a non-monomial function z^q of some complex coordinate z with a nonpositive integer q is given, z^q should be Taylor expanded around the some non-zero point $p \in \mathbb{C}$ to express it as a twisted Fock algebra:

$$z^q = p^q + qp^{q-1}(z-p) + \frac{q(q-1)}{2}p^{q-2}(z-p)^2 + \dots \quad (6.21)$$

In the case that the radius of convergence of this expansion is not enough to cover the whole of U_b , we have to divide U_b into smaller ones, $U_b = \cup_{b_i} U_{b_i}$ and choose proper points for the Taylor expansions on each U_{b_i} , to make each expansions converge. For the higher dimensional $\mathbb{C}P^N$ we can use a similar procedure to the one dimensional case, and the twisted Fock algebra for $\mathbb{C}P^N$ is derived.

To avoid such kind of complications concerning $(z_b^a)^{1/\hbar-|m|}$ and $(\bar{z}_b^a)^{1/\hbar-|n|}$, we can introduce a slightly different representation from the above twisted Fock representation of $\mathbb{C}P^N$. Let us consider the case that the noncommutative parameter is the following value,

$$1/\hbar = L \in \mathbb{Z}, \quad L \geq 0, \quad (6.22)$$

Then, we define F_a^L on U_a as a subspace of a local twisted Fock algebra F_{U_a} ,

$$F_a^L = \left\{ \sum_{\vec{m}, \vec{n}} A_{\vec{m}\vec{n}} |\vec{m}\rangle_{aa} \langle \vec{n}| \mid A_{\vec{m}\vec{n}} \in \mathbb{C}, \quad |m| \leq L, \quad |n| \leq L \right\}. \quad (6.23)$$

The bases on U_a are related to those on U_b as,

$$\sqrt{\frac{(L-|n|)!}{(L-|m|)!}} |\vec{m}\rangle_{aa} \langle \vec{n}| = \sqrt{\frac{(L-|n'|)!}{(L-|m'|)!}} |\vec{m'}\rangle_{bb} \langle \vec{n'}|, \quad (6.24)$$

where

$$\vec{m'} = (m_0, \dots, m_{a-1}, L-|m|, m_{a+1}, \dots, m_{b-1}, m_{b+1}, \dots, m_N), \quad (6.25)$$

$$\vec{n'} = (n_0, \dots, n_{a-1}, L-|n|, n_{a+1}, \dots, n_{b-1}, n_{b+1}, \dots, n_N). \quad (6.26)$$

Using the expression of (6.24), we can define $|\vec{n}\rangle_{aa}\langle\vec{n}|$ on the whole of U_b . Therefore, the operators in F_a^L can be extended to the whole of $\mathbb{C}P^N$ by using the relation like (6.24).

Under the condition (6.22), the creation and annihilation operators on F_{U_a} is changed from the definition (3.1). Similarly to (3.49) and (3.50), let us define a creation operator $a_{a,i}^{L\dagger}$ and an annihilation operator $\underline{a}_{a,i}^L$ restricted on F_a^L by

$$a_{a,i}^{L\dagger} = \sum_{0 \leq |n| \leq L-1} \sqrt{n_i + 1} |\vec{n} + \vec{e}_i\rangle_{aa} \langle \vec{n}| = z_a^i \left[1 - \left(\frac{|z_a|^2}{1 + |z_a|^2} \right)^L \right], \quad (6.27)$$

$$\underline{a}_{a,i}^L = \sum_{0 \leq |n| \leq L-1} \sqrt{n_i + 1} |\vec{n}\rangle_{aa} \langle \vec{n} + \vec{e}_i| = L \frac{\bar{z}_a^i}{1 + |z_a|^2}. \quad (6.28)$$

By the restriction on F_a^L , $a_{a,i}^{L\dagger}$ is shifted from z_a^i . These operators satisfy the following commutation relation,

$$\begin{aligned} [a_{a,i}^L, a_{a,j}^{L\dagger}] &= \delta_{ij} \left(\sum_{0 \leq |n| \leq L} |\vec{n}\rangle_{aa} \langle \vec{n}| - \sum_{|n|=L} (n_i + 1) |\vec{n}\rangle_{aa} \langle \vec{n}| \right) \\ &= \delta_{ij} - \delta_{ij} \left(\frac{|z_a|^2}{1 + |z_a|^2} \right)^L \left(1 + L \frac{|z_a^i|^2}{|z_a|^2} \right). \end{aligned} \quad (6.29)$$

Example 4 : Fock representation of noncommutative $\mathbb{C}H^N$

Here, we give an explicit expression of the Fock representation of noncommutative of $\mathbb{C}H^N$ [22, 23].

We choose a Kähler potential satisfies the condition (3.14)

$$\Phi = -\ln(1 - |z|^2), \quad (6.30)$$

where $|z|^2 = \sum_i^N |z^i|^2$. A star product is given as follows [22, 23]:

$$f * g = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1 \bar{k}_1} \cdots g_{j_n \bar{k}_n} (D^{j_1} \cdots D^{j_n} f) D^{\bar{k}_1} \cdots D^{\bar{k}_n} g, \quad (6.31)$$

where

$$c_n(\hbar) = \frac{\Gamma(1/\hbar)}{n! \Gamma(n + 1/\hbar)}, \quad D^{\bar{i}} = g^{\bar{i}j} \partial_j, \quad D^i = g^{i\bar{j}} \partial_{\bar{j}}. \quad (6.32)$$

The creation and annihilation operators are given as

$$a_i^\dagger = z^i, \quad \underline{a}_i = \frac{1}{\hbar} \partial_i \Phi = \frac{\bar{z}^i}{\hbar(1 - |z|^2)}, \quad a_i = \bar{z}^i, \quad \underline{a}_i^\dagger = \frac{1}{\hbar} \partial_i \Phi = \frac{z^i}{\hbar(1 - |z|^2)}. \quad (6.33)$$

and a vacuum is

$$|\vec{0}\rangle\langle\vec{0}| = e^{-\Phi/\hbar} = (1 - |z|^2)^{1/\hbar}. \quad (6.34)$$

Bases of the Fock representation on $\mathbb{C}H^N$ are constructed as

$$\begin{aligned} |\vec{m}\rangle\langle\vec{n}| &= \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (a^\dagger)_*^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_*^{\vec{n}} \\ &= \frac{1}{\sqrt{\vec{m}!\vec{n}!\hbar^{|\vec{n}|}}} (z)_*^{\vec{m}} * e^{-\Phi/\hbar} * (\partial\Phi)_*^{\vec{n}}. \end{aligned} \quad (6.35)$$

By using (3.28), (3.29) and the following relation which is shown in [22],

$$(\partial\Phi)_*^{\vec{n}} = \frac{(-\hbar)^{|\vec{n}|} \Gamma(1/\hbar + |\vec{n}|)}{\Gamma(1/\hbar)} \left(\frac{\bar{z}}{1 - |z|^2} \right)^{\vec{n}}, \quad (6.36)$$

the bases can be explicitly written as

$$|\vec{m}\rangle\langle\vec{n}| = \frac{(-1)^{|\vec{n}|} \Gamma(1/\hbar + |\vec{n}|)}{\sqrt{\vec{m}!\vec{n}!\Gamma(1/\hbar)}} (z)^{\vec{m}} (\bar{z})^{\vec{n}} (1 - |z|^2)^{1/\hbar}. \quad (6.37)$$

These are defined globally. For $\mathbb{C}H^N$, trace density is given by the usual Riemannian volume form

$$\mu_g = \frac{1}{(1 - |z|^2)^{N+1}}. \quad (6.38)$$

Therefore,

$$\begin{aligned} c_0 &= \int_{\mathbb{C}H^N} dz^{2D} \mu_g |0\rangle\langle 0| \\ &= \int_{\mathbb{C}H^N} dz^{2D} \frac{1}{(1 - |z|^2)^{\frac{1}{\hbar} - (N+1)}} \\ &= \pi^N \frac{\Gamma(1/\hbar - N)}{\Gamma(1/\hbar)}, \end{aligned} \quad (6.39)$$

and the trace is given by the integration

$$\text{Tr}_{\mathbb{C}H^N} |\vec{m}\rangle\langle\vec{n}| = \frac{\Gamma(1/\hbar)}{\pi^N \Gamma(1/\hbar - N)} \int_{\mathbb{C}H^N} dz^{2D} \mu_g |\vec{m}\rangle\langle\vec{n}| = \delta_{\vec{m}\vec{n}}. \quad (6.40)$$

At the end of this section, we mention a special class of Kähler manifolds. The above examples, \mathbb{C}^N , cylinder, $\mathbb{C}P^N$ and $\mathbb{C}H^N$, have Kähler potentials which depend only on the absolute values of complex coordinates:

$$\Phi(z, \bar{z}) = \tilde{\Phi}(|z_1|, |z_2|, \dots, |z_N|). \quad (6.41)$$

For this case, we obtain the usual Fock algebra by the following proposition.

Proposition 6.1. *When a Kähler potential is an analytic function and has the form of (6.41), $|\vec{0}\rangle\langle\vec{0}| * (a^\dagger)_*^{\vec{m}}$ and $(a^\dagger)_*^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}|$ are equal to $|\vec{0}\rangle\langle\vec{0}| * (\underline{a})_*^{\vec{m}}$ and $(\underline{a})_*^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}|$ up to a constant, respectively.*

Proof. From the identity $L_{\partial_i \Phi} = \hbar e^{-\Phi/\hbar} \partial_i e^{\Phi/\hbar}$,

$$\begin{aligned} & |\vec{0}\rangle\langle\vec{0}| * (\partial_1 \Phi)_*^{n_1} * \dots * (\partial_{i_N} \Phi)_*^{n_N} \\ &= \hbar^{|\vec{n}|} |\vec{0}\rangle\langle\vec{0}| * \left(\bar{z}_1^{n_1} \dots \bar{z}_N^{n_N} e^{-\tilde{\Phi}/\hbar} \left(\frac{\partial}{\partial |z_1|} \right)^{n_1} \dots \left(\frac{\partial}{\partial |z_N|} \right)^{n_N} e^{\tilde{\Phi}/\hbar} \right). \end{aligned}$$

By using Lemma 3.4, $e^{-\Phi/\hbar} * f(z, \bar{z}) = e^{-\Phi/\hbar} * f(0, \bar{z})$,

$$e^{-\tilde{\Phi}/\hbar} \left(\frac{\partial}{\partial |z_1|} \right)^{n_1} \dots \left(\frac{\partial}{\partial |z_N|} \right)^{n_N} e^{\tilde{\Phi}/\hbar}$$

in the above equation can be replaced by a constant, which we here denote by $C(\vec{n})$. Then

$$|\vec{0}\rangle\langle\vec{0}| * (\partial_1 \Phi)_*^{n_1} * \dots * (\partial_{i_N} \Phi)_*^{n_N} = \hbar^{|\vec{n}|} C(\vec{n}) |\vec{0}\rangle\langle\vec{0}| * (\bar{z})_*^{\vec{n}}. \quad (6.42)$$

□

As a corollary we obtain the following.

Corollary 6.2. *When a Kähler potential is an analytic function and has the form of (6.41), $|\vec{m}\rangle\langle\vec{n}| = \hbar^{|\vec{n}|} C(\vec{n}) |\vec{m}\rangle\langle\vec{n}|$. Here $C(\vec{n}) = e^{-\tilde{\Phi}/\hbar} \left(\frac{\partial}{\partial |z_1|} \right)^{n_1} \dots \left(\frac{\partial}{\partial |z_N|} \right)^{n_N} e^{\tilde{\Phi}/\hbar} |_{z=0}$.*

This corollary is also shown by using the definition of $H_{\vec{m}, \vec{n}}$ in (3.31) without Proposition 6.1. For a Kähler potential $\tilde{\Phi}$, $H_{\vec{m}, \vec{n}}$ is proportional to $\delta_{\vec{m}, \vec{n}}$,

$$e^{\tilde{\Phi}/\hbar} = \sum_{\vec{m}, \vec{n}} \frac{C(\vec{n})}{\vec{n}!} \delta_{\vec{m}, \vec{n}}(z)^{\vec{m}} (\bar{z})^{\vec{n}}, \quad (6.43)$$

$$H_{\vec{m}, \vec{n}} = \frac{C(\vec{n})}{\vec{n}!} \delta_{\vec{m}, \vec{n}}. \quad (6.44)$$

Here $C(\vec{n})$ is a constant and is given as

$$C(\vec{n}) = \left(\frac{\partial}{\partial |z_1|} \right)^{n_1} \cdots \left(\frac{\partial}{\partial |z_N|} \right)^{n_N} e^{\tilde{\Phi}/\hbar}|_{z=0}. \quad (6.45)$$

By using (3.34), we find

$$|\vec{m}\rangle \langle \underline{\vec{n}}| = C(\vec{n}) |\vec{m}\rangle \langle \vec{n}|. \quad (6.46)$$

7 Summary

Twisted Fock representations of general noncommutative Kähler manifolds are constructed. The noncommutative Kähler manifolds studied in this article are given by deformation quantization with separation of variables. Using this type of deformation quantization, the twisted Fock representation which constructed based on two sets of creation and annihilation operators was introduced with the concrete expressions of them on a local coordinate chart. The corresponding functions are given by the local complex coordinates, the Kähler potentials and partial derivatives of them with respect to the coordinates. The dictionary to translate bases of the twisted Fock representation into functions is given as table 1. They are defined on a local coordinate chart, and they are extended by the transition functions given in Section 4. This extension is achieved by essentially the result that the star products with separation of variables have a trivial transition function. We also gave examples of the twisted Fock representation of Kähler manifolds, \mathbb{C}^N , cylinder, $\mathbb{C}P^N$ and $\mathbb{C}H^N$. The trace operation as an integration over a manifold is obtained by traces of matrix representations for the \mathbb{C}^N and $\mathbb{C}H^N$.

Acknowledgments

A.S. was supported in part by JSPS KAKENHI Grant Number 16K05138.

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